Teaching modes of reasoning: Redesigning the *Art of Approximation in Science and Engineering*

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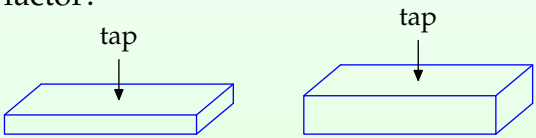
Modes of reasoning or topics?

Doubling the block's thickness changes the note frequency by what factor?



a. 2
b. $\sqrt{2}$
c. 1
d. $1/\sqrt{2}$
e. $1/2$

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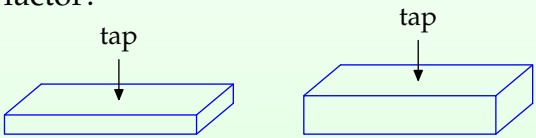

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A spring model explains the doubling in frequency

Compare the stored energies for the same deflection y:

stored energy $\sim \underbrace{\text{stiffness}}_{k} \times y^2$.

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Compare the stored energies for the same deflection y:

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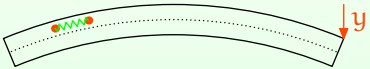

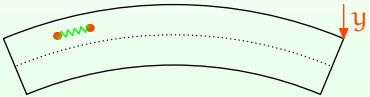

$4\times$ the energy per spring

A spring model explains the doubling in frequency

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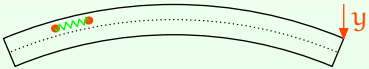

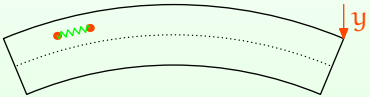

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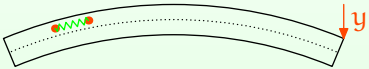

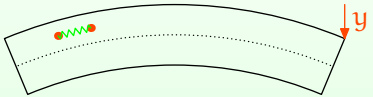

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$8\times$ $\underbrace{\text{the stored energy}}_{\text{stiffness}\times y^2}$

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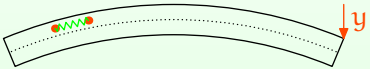
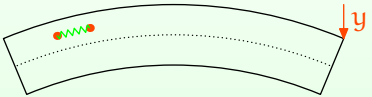




$4\times$ the energy per spring

$2\times$ the number of springs

$8\times$ $\underbrace{\text{the stored energy}}_{\text{stiffness}\times y^2}$

(bending) frequency $\sim \sqrt{\dfrac{\text{stiffness}}{\text{mass}}} = \sqrt{\dfrac{8\times}{2\times}} = 2\times.$

Modes of reasoning or topics?

Modes of reasoning are a better organization than topics

Using modes of reasoning makes the course finite

Using modes of reasoning promotes transfer

Using modes of reasoning promotes long-lasting learning

Topics are many, life is short

sound
waves
mechanical properties
thermal properties
weather
fluid drag
turbulence

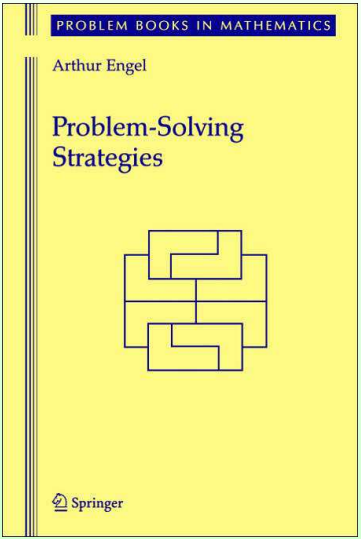
gravitation
prime numbers
retinal rod
biomechanics
astrophysics
financial math
. . .

Where do you stop?

Teaching in Cambridge, England, I moved toward modes of reasoning only subconsciously

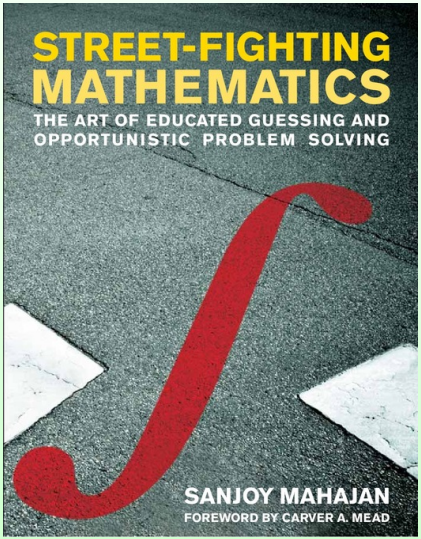
I was influenced by problem solving in mathematics

The Invariance Principle
Coloring Proofs
The Extremal Principle
The Box Principle
Enumerative Combinatorics
Number Theory
Inequalities
The Induction Principle
Sequences
Polynomials
Functional Equations
Geometry
Games
Further Strategies



I used it for *Street-Fighting Mathematics*

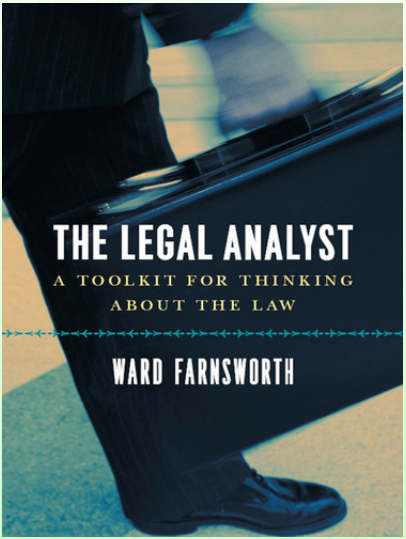
Dimensions
Easy cases
Lumping
Pictorial proofs
Taking out the big part
Analogy



Modes of reasoning now seemed to appear everywhere

Pt. I. Incentives
- Ex ante and ex post
- The idea of efficiency
- Thinking at the margin
- The single owner
- The least cost avoider
- Administrative cost
- Rents
- The Coase theorem

. . .



Modes of reasoning for science and engineering organized themselves slowly

Easy cases
Divide and conquer
Spring models
Lumping
Proportional reasoning
Symmetry/conservation
Abstraction
Probabilistic reasoning
Dimensional analysis

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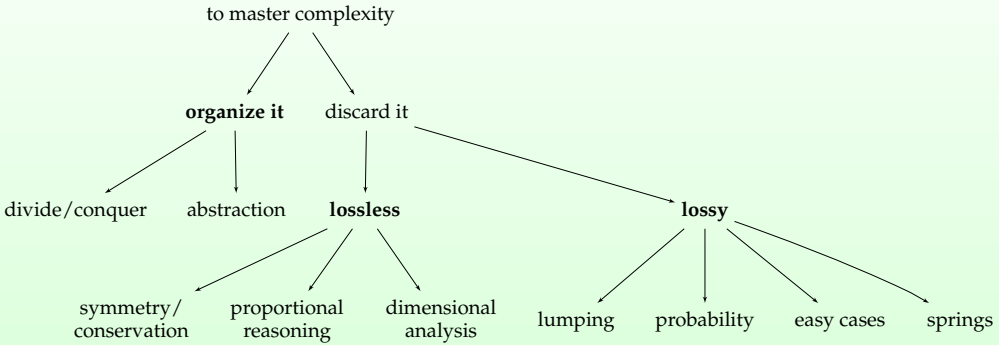
Modes of reasoning for science and engineering organized themselves slowly

Organizing	*Discarding*
Abstraction	Proportional
Divide and conquer	Symmetry/conservation
	Dimensional analysis
	Probabilistic
	Easy cases
	Spring models
	Lumping

Modes of reasoning for science and engineering organized themselves slowly

Organizing	*Discard: lossless*	*Discard: lossy*
Abstraction	Proportional	Probabilistic
Divide and conquer	Symmetry/conservation	Easy cases
	Dimensional analysis	Spring models
		Lumping

Modes of reasoning for science and engineering organized themselves around mastering complexity



Using modes of reasoning makes the course finite

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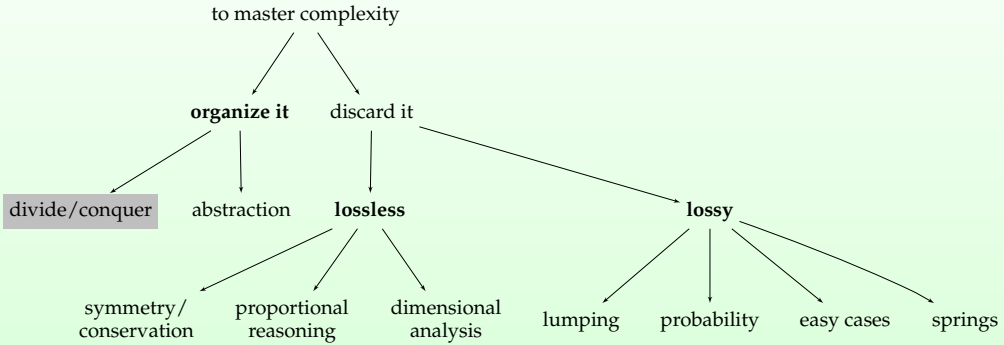
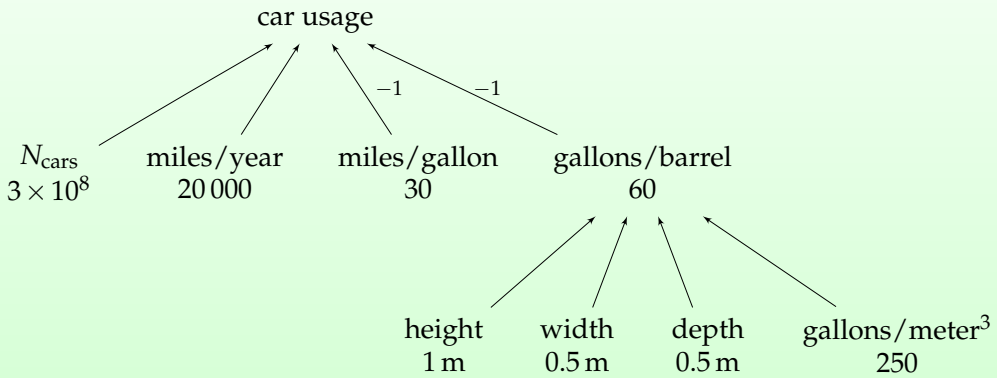
The tree gives each mode of reasoning a place



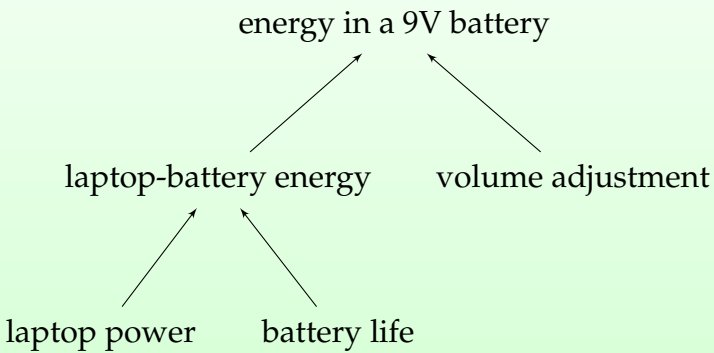
Each mode of reasoning contains examples: Divide and conquer

How many barrels of oil does the United States import in a year?



Each mode of reasoning contains examples: Divide and conquer

How much energy does a 9-volt battery contain?



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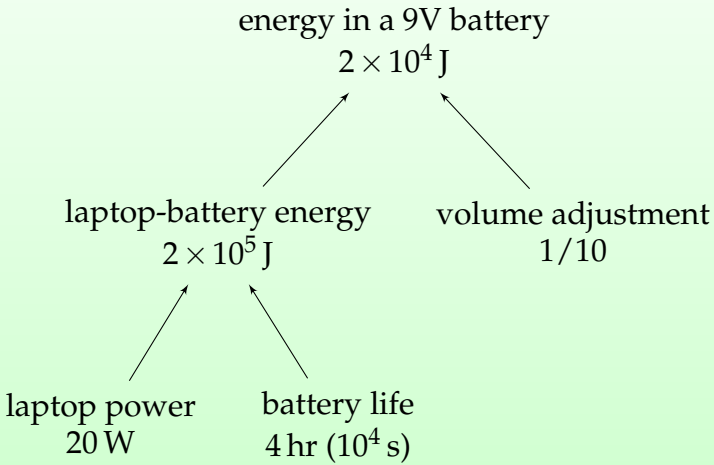
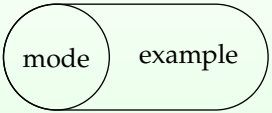
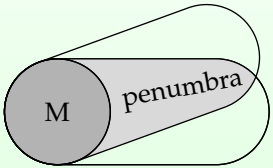


Using modes of reasoning promotes transfer



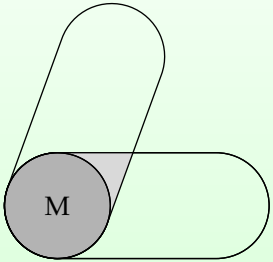
Using modes of reasoning promotes transfer

When teaching by topics, it is too easy to use too-similar examples.



Using modes of reasoning promotes transfer

Diverse examples help clarify the core idea.



Using modes of reasoning automatically produces diverse examples.

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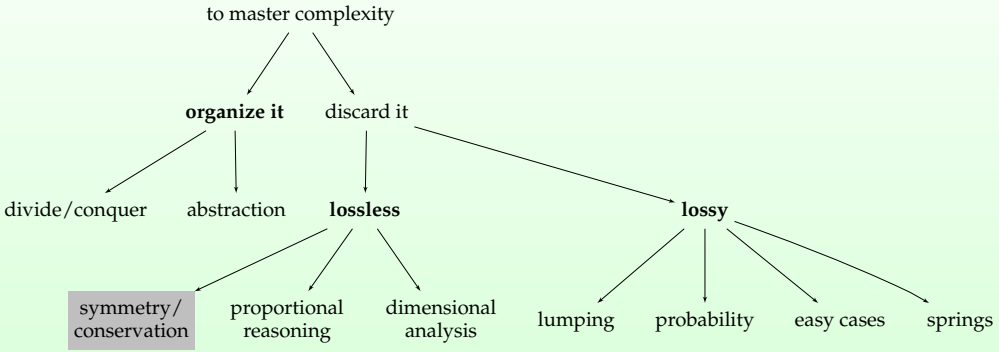
Using modes of reasoning makes the course finite

Using modes of reasoning promotes transfer

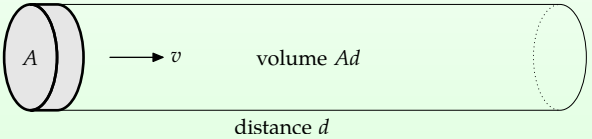
Using modes of reasoning promotes long-lasting learning

The tree gives each mode of reasoning a place

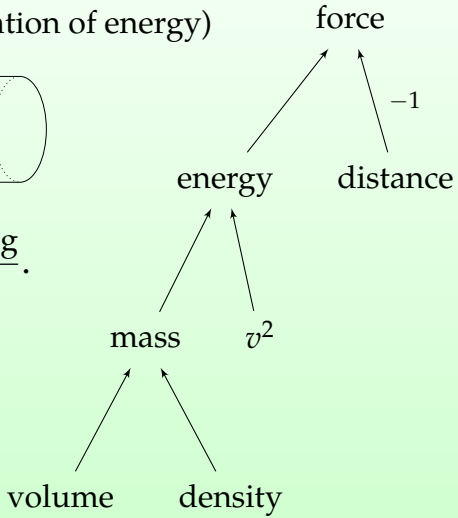


Each mode of reasoning contains examples: Symmetry and conservation

How fast will the cone fall? (by conservation of energy)



$$\text{drag force} = \frac{\text{energy consumed by drag}}{\text{distance traveled}}.$$



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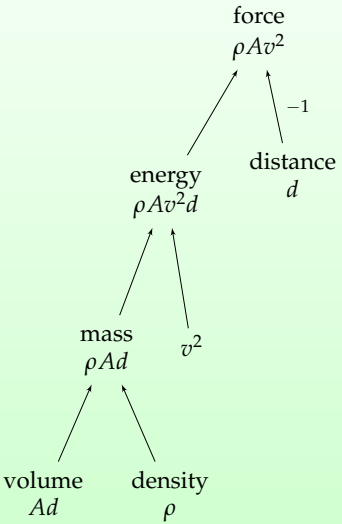

Each mode of reasoning contains examples: Symmetry and conservation

How fast will the cone fall?

$$F \sim \rho A v^2$$

$$v \sim \sqrt{\frac{F}{\rho A}} \sim \sqrt{\frac{10^{-3}\,\mathrm{kg} \times 10\,\mathrm{m/s^2}}{1\,\mathrm{kg/m^3} \times 0.01\,\mathrm{m^2}}} \sim 1\,\mathrm{m/s}.$$

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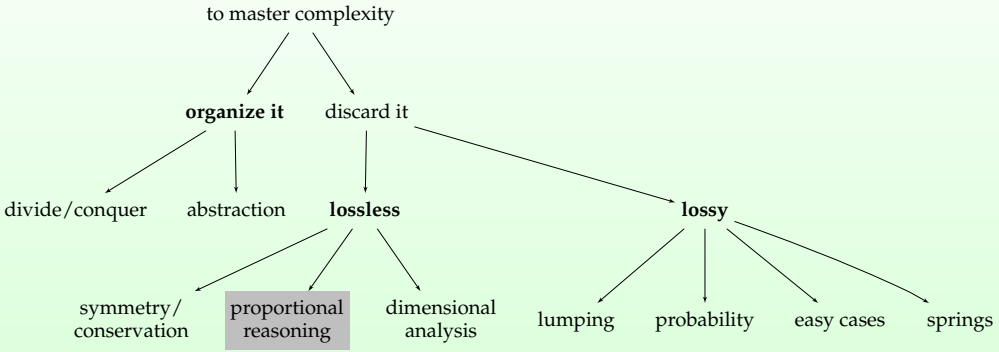
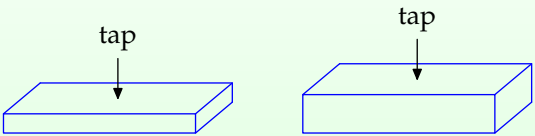


Each mode of reasoning contains examples: Proportional reasoning

Wood blocks



frequency $\propto$ thickness$^{?}$

Each mode of reasoning contains examples: Proportional reasoning

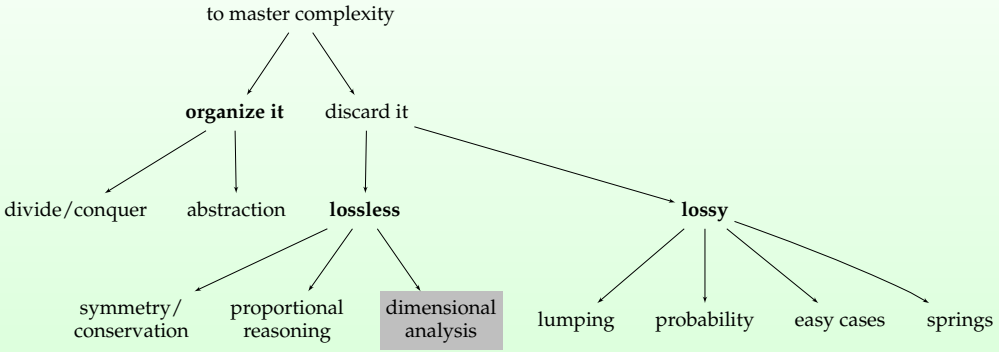
Falling cones again

$$\frac{v_{\text{four stacked cones}}}{v_{\text{one stacked cone}}} = \begin{cases} 4 \\ 2 \\ or \\ \sqrt{2} \end{cases}$$

Equivalently,

$v \propto (\text{number of cones})^{?}$

The tree gives each mode of reasoning a place



Each mode of reasoning contains examples: Dimensional analysis

Motto: *The uncompared quantity is not worth knowing.*

$$\frac{\text{cost of 9-volt battery energy}}{\text{cost of line (mains) power}}$$

Each mode of reasoning contains examples: Dimensional analysis

Motto: *The uncompared quantity is not worth knowing.*

$$\frac{\text{cost of 9-volt battery energy}}{\text{cost of line (mains) power}} \sim \frac{\$1 \quad / \quad 2 \times 10^4\,\text{J}}{\$0.15 \,/\, 3.6 \times 10^6\,\text{J}}$$
$$\approx 7 \times 180$$
$$\sim 1000.$$

Each mode of reasoning contains examples: Lumping

Every number is of the form:

$$\begin{pmatrix} \text{one} \\ \textit{or} \\ \text{few} \end{pmatrix} \times 10^{n},$$

where

$\text{few}^2 = 10.$

Each mode of reasoning contains examples: Lumping

How many seconds in a year?

$$\frac{365\,\text{days}}{\text{year}} \times \frac{24\,\text{hours}}{\text{day}} \times \frac{3600\,\text{seconds}}{\text{hour}}$$

Each mode of reasoning contains examples: Lumping

How many seconds in a year?

$$\frac{\text{few} \times 10^2 \text{ days}}{\text{year}} \times \frac{24 \text{ hours}}{\text{day}} \times \frac{3600 \text{ seconds}}{\text{hour}}$$

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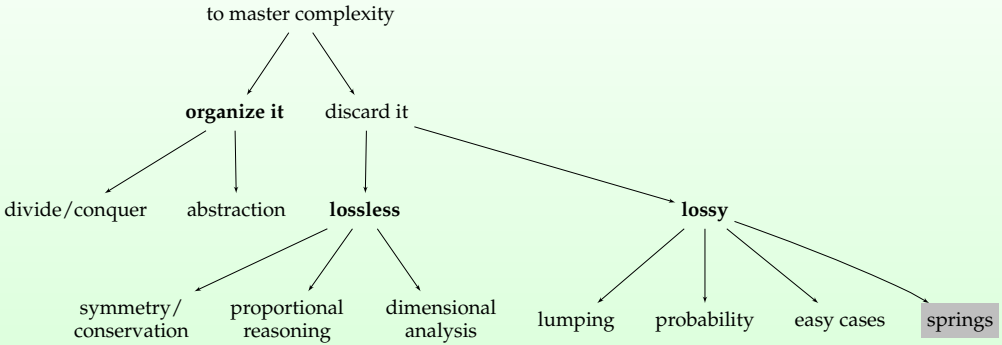
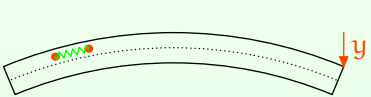
$$\sim \text{few} \times 10^7 \frac{\text{seconds}}{\text{year}}$$

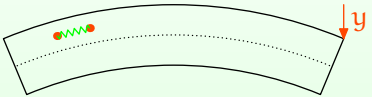
The tree gives each mode of reasoning a place



Each mode of reasoning contains examples: Springs

Wood blocks



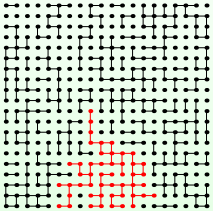


Each mode of reasoning contains examples: Springs

Why is the sky blue?

How much energy does the earth–sun system lose in gravitational radiation?

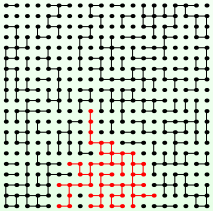
Connections are more important than facts alone



big cluster = 12%

$p_{bond} = 0.40$

Connections are more important than facts alone



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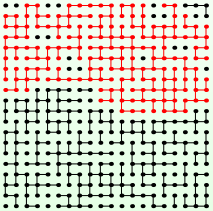

big cluster = 42%

$p_{bond} = 0.50$

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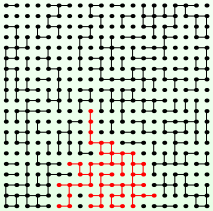

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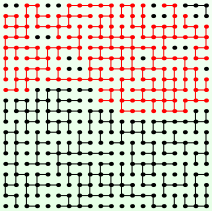

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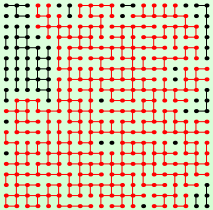
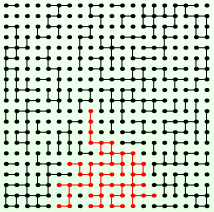


big cluster = 81%

$p_{bond} = 0.55$

Connections are more important than facts alone



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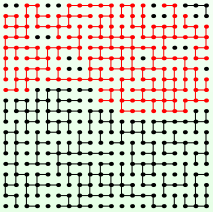

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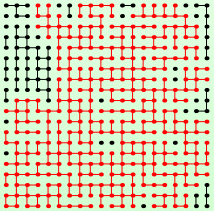

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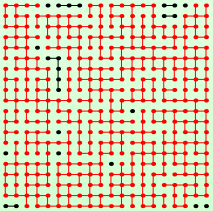

big cluster = 94%

$p_{bond} = 0.60$

Using modes of reasoning promotes long-lasting learning

Modes of reasoning are a better organization than topics

Using modes of reasoning makes the course finite

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Modes of reasoning are a better organization than topics

The goal [of teaching] should be, not to implant in the students' mind every fact that the teacher knows now;

but rather to implant a *way of thinking* that enables the student, in the future, to learn in one year what the teacher learned in two years.

Only in that way can we continue to advance from one generation to the next.

—*Edwin T. Jaynes (1922–1998)*

Teaching modes of reasoning: Redesigning the *Modes of Reasoning in Science and Engineering*

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